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Technical Note TN 15

A note on the zetafunction of Riemann-Hurwitz

by

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1. Introduction

In this note some old and new results concerning Hurwitz's generalization of the zetafunction of Riemann^{*)}

$$(1.1) \quad \zeta(s, a) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}$$

will be discussed.

For simplicity, we shall suppose that $a > 0$. The definition (1.1) holds for $\text{Re } s > 1$ but the following analytic continuation can easily be obtained

$$(1.2) \quad \zeta(s, a) = \frac{\Gamma(1-s)}{2\pi i} \int_{-\infty}^{0^+} \frac{z^{s-1} e^{az}}{1-e^z} dz.$$

Hence $\zeta(s, a)$ has a simple pole at $s=1$ with the residue $+1$.

We note the following simple relation

$$(1.3) \quad \zeta(s, a) = \zeta(s, a+m) + \sum_{n=0}^{m-1} (n+a)^{-s},$$

where m is a positive integer. This so-called shift-rule can be successfully applied in many numerical applications.

We shall prove the following well-known asymptotic expansion

$$(1.4) \quad \Gamma(s) \zeta(s, a) = \sum_{n=0}^N \frac{B_n \Gamma(n+s-1)}{n! (a-1)^{n+s-1}} + R_N,$$

where N is even. The remainder is less in absolute value than the first omitted term. We note that the Bernoullian coefficients B_n in (1.4) are defined by the generating function

$$(1.5) \quad \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n.$$

The expansion (1.4) is valid for all s and a with the exception of course of $s=1$. The practical applicability of (1.4) can be greatly enhanced by using the shift rule. We have by combining (1.3) and (1.4)

*) Whittaker and Watson. Modern Analysis. 13.11.

$$(1.6) \quad \Gamma(s) \zeta(s, a) = \Gamma(s) \sum_{n=0}^{m-1} (n+a)^{-s} + \sum_{n=0}^{\infty} \frac{B_n \Gamma(n+s-1)}{n! (m+a-1)^{n+s-1}},$$

where m is an arbitrary integer.

Next we shall prove the convergent expansion of $\zeta(s, a)$ in a faculty series. This result is probably new.

$$(1.7) \quad \Gamma(s) \zeta(s, a) = \sum_{n=0}^{\infty} c_n B(a, n+s-1),$$

where the coefficients c_n are defined by their generating function

$$(1.8) \quad \left\{ -u^{-1} \ln(1-u) \right\}^{s-1} = \sum_{n=0}^{\infty} c_n u^n.$$

The first few coefficients are

$$(1.9) \quad c_0 = 1 \quad c_1 = \frac{1}{2}(s-1) \quad c_2 = \frac{1}{24}(s-1)(3s+2).$$

They can be calculated from the recurrent relation

$$(1.10) \quad n c_n = \sum_{k=0}^{n-1} \frac{n(s-1)-sk}{n-k+1} c_k.$$

Their asymptotic behaviour is

$$(1.11) \quad c_n = \frac{s-1}{n+s-1} \left\{ \ln(n+s-1) \right\}^{s-2} \left\{ 1 + O(\ln^{-2} n) \right\}.$$

We note that for $s=2$ the c_n are very simple, viz.

$$(1.12) \quad c_n = (n+1)^{-1}, \quad s=2.$$

The series (1.7) converges for all values of s , again with $s \neq 1$.

The case with which the generalized zetafunction can be computed by using either (1.6) or (1.7) gives the possibility of a simple and accurate computation of the two functions

$$(1.13) \quad C(x, s) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \frac{\cos 2n\pi x}{n^s}, \quad 0 \leq x \leq 1,$$

and

$$(1.14) \quad S(x, s) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \frac{\sin 2n\pi x}{n^s}, \quad 0 \leq x \leq 1,$$

if s is a small positive constant.

We shall write

$$(1.15) \quad F(x, s) \stackrel{\text{def}}{=} C(x, s) + i S(x, s).$$

Then it follows from a result of Hurwitz ^{*)} that

*) Whittaker and Watson l.c. 13.15.

$$(1.16) \quad F(x,s) = \frac{\Gamma(1-s)}{(2\pi)^{1-s}} \left\{ e^{\frac{1}{2}\pi(1-s)i} \zeta(1-s, x) + e^{-\frac{1}{2}\pi(1-s)i} \zeta(1-s, 1-x) \right\}$$

valid for $0 < x < 1$.

If e.g. the asymptotic expansion (1.6) is applied upon (1.16) we obtain

$$(1.17) \quad C(x,s) = (2\pi)^{s-1} \Gamma(1-s) \sin \frac{1}{2}\pi s \left\{ \sum_{n=-m}^{n=m-1} |n+x|^{s-1} + \right. \\ \left. + \sum_{n=0}^{\infty} \frac{B_n}{n!} \frac{\Gamma(n-s)}{\Gamma(1-s)} \left\{ (m-1+x)^{s-n} + (m-x)^{s-n} \right\} \right\}$$

and a similar result for $S(x,s)$.

If s is a positive integer the result follows by a simple limit operation.

Finally a simple proof will be given of two expansions due to Lindelöf

$$(1.18) \quad C(x,s) = (2\pi)^{s-1} \Gamma(1-s) \sin \frac{1}{2}\pi s \left\{ x^{s-1} + 2 \sum_{j=0}^{\infty} \binom{s-1}{2j} \zeta(2j+1-s) x^{2j} \right\}$$

and

$$(1.19) \quad S(x,s) = (2\pi)^{s-1} \Gamma(1-s) \cos \frac{1}{2}\pi s \left\{ x^{s-1} + 2 \sum_{j=0}^{\infty} \binom{s-1}{2j+1} \zeta(2j+2-s) x^{2j+1} \right\}$$

which are convergent for $0 < x < 1$.

2. From (1.1) it follows that for $\text{Re } s > 1$

$$(2.1) \quad \Gamma(s) \zeta(s, a) = \int_0^{\infty} \frac{e^{-at}}{1-e^{-t}} t^{s-1} dt.$$

Since

$$(2.2) \quad \frac{t}{e^t - 1} = \sum_{n=0}^N \frac{B_n}{n!} t^n + \Theta \frac{B_{N+2}}{(N+2)!} t^{N+2}$$

for even N and $|\Theta| < 1$, we obtain

$$\Gamma(s) \zeta(s, a) = \sum_{n=0}^N \frac{B_n}{n!} \int_0^{\infty} e^{-at} t^{n+s-2} dt + \text{remainder},$$

from which (1.4) easily follows.

If in (2.1) the substitution $t = -\ln(1-u)$ is made we obtain

$$(2.3) \quad \Gamma(s) \zeta(s, a) = \int_0^1 (1-u)^{a-1} u^{s-2} f(u) du,$$

where

$$(2.4) \quad f(u) = \left\{ -\frac{1}{u} \ln(1-u) \right\}^{s-1}.$$

If $f(u)$ is expanded in the power series

$$(2.5) \quad f(u) = \sum_{n=0}^{\infty} c_n u^n,$$

where

$$c_0 = 1 \quad c_1 = \frac{1}{2}(s-1) \quad c_2 = \frac{1}{24}(s-1)(3s+2) \quad \text{etc.}$$

we obtain a convergent expansion of the right-hand side of (2.3) in a faculty series viz.

$$(2.6) \quad \Gamma(s) \zeta(s, a) = \sum_{n=0}^{\infty} c_n B(a, n+s-1),$$

or

$$(2.7) \quad \zeta(s, a) = \frac{\Gamma(a)}{\Gamma(a+s)} \left\{ \frac{a+s-1}{s-1} + c_1 + \frac{s}{a+s} c_2 + \frac{s(s+1)}{(a+s)(a+s+1)} c_3 + \dots \right\}.$$

The recurrent relation (1.10) can be derived in the following way. Logarithmic differentiation of $u^{s-1}f(u)$ yields

$$\left\{ -\ln(1-u) \right\} \frac{d}{du} (u^{s-1}f) = (s-1)(1-u)^{-1}(u^{s-1}f).$$

Substitution of (2.5) gives the identity

$$\sum_{j=1}^{\infty} j^{-1} u^j \sum_{k=0}^{\infty} (k+s-1) c_k u^{k+s-2} = (s-1) \sum_{j=1}^{\infty} u^j \sum_{k=0}^{\infty} c_k u^{k+s-2}.$$

Comparison of the coefficient of equal powers on both sides leads at once to the relation (1.10).

The asymptotic behaviour (1.11) of c_n can be found from Cauchy's formula

$$(2.8) \quad c_n = \frac{1}{2\pi i} \oint z^{-s-n} \{-\ln(1-z)\}^{s-1} dz,$$

where the path of integration is a small circle around the origin. The integrand has a branchpoint at $z=1$ and the contour can be deformed into a path along the upper and lower sides of the cut from 1 to $+\infty$. This gives

$$(2.9) \quad \pi c_n = \text{Im} \int_0^\infty e^{-(n+s-1)t} \{\pi i - \ln(e^t - 1)\}^{s-1} dt.$$

The asymptotic behaviour of the right-hand side is determined by the nature of $\{\pi i - \ln(e^t - 1)\}^{s-1}$ at the origin.

Since

$$\{\pi i - \ln(e^t - 1)\}^{s-1} = \left(\ln \frac{1}{t}\right)^{s-1} \left\{1 + \frac{(s-1)\pi i}{\ln 1/t} + O(\ln^{-2} \frac{1}{t})\right\}$$

we have asymptotically

$$(2.10) \quad c_n \approx \frac{(s-1)\{\ln(n+s-1)\}^{s-2}}{n+s-1}.$$

For $s=2$ the right-hand side of (2.10) gives also the right expression.

An amusing particular case of (2.6) is obtained for $s=2$ and $a=m$ =positive integer. It follows easily that

$$(2.11) \quad \sum_{n=m}^{\infty} \frac{1}{n^2} = (m-1)! \sum_{k=0}^{\infty} \frac{1}{(k+1)^2(k+2)\dots(k+m)}.$$

3. Quoting Hurwitz's result *)

$$(3.1) \quad \zeta(s, a) = 2(2\pi)^{s-1} \Gamma(1-s) \left\{ \sin \frac{1}{2}\pi s C(a, 1-s) + \cos \frac{1}{2}\pi s S(a, 1-s) \right\},$$

it can easily be deduced that for $0 < x < 1$

$$(3.2) \quad C(x, s) = (2\pi)^{s-1} \Gamma(1-s) \sin \frac{1}{2}\pi s \left\{ \zeta(1-s, x) + \zeta(1-s, 1-x) \right\},$$

and

$$(3.3) \quad S(x, s) = (2\pi)^{s-1} \Gamma(1-s) \cos \frac{1}{2}\pi s \left\{ \zeta(1-s, x) - \zeta(1-s, 1-x) \right\}.$$

These two formulae may be combined into the single relation (1.16).

From

$$(3.4) \quad F(x, s) = C(x, s) + i S(x, s) = \sum_{n=1}^{\infty} n^{-s} e^{2n\pi xi}$$

it follows that for any positive integer m

$$(3.5) \quad \sum_{j=0}^{m-1} F\left(x + \frac{j}{m}, s\right) = m^{1-s} F(mx, s).$$

Further we have

$$(3.6) \quad F(1-x, s) = F^*(x, s),$$

where F^* denotes the complex conjugate of F .

For $m=2$ it follows from (3.5) by using (3.6) that

$$(3.7) \quad F(x, s) = 2^{s-1} \left\{ F\left(\frac{1}{2}x, s\right) + F^*\left(\frac{1}{2}-\frac{1}{2}x, s\right) \right\}.$$

This means that it is sufficient to calculate $F(x, s)$ for x in the interval $(0, 1/3)$ only.

Next a simple proof will be given of Lindelöf's expansions (1.18) and (1.19). We have for $0 < a < 1$

$$\begin{aligned} \zeta(s, a) &= a^{-s} + \sum_{n=1}^{\infty} n^{-s} \left(1 + \frac{a}{n}\right)^{-s} = \\ &= a^{-s} + \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} \binom{-s}{j} a^j n^{-s-j} = \\ &= a^{-s} + \sum_{j=0}^{\infty} \binom{-s}{j} a^j \sum_{n=1}^{\infty} n^{-s-j} \end{aligned}$$

so that

$$(3.8) \quad \zeta(s, a) = a^{-s} + \sum_{j=0}^{\infty} \binom{-s}{j} \zeta(j+s) a^j.$$

In a similar way we have

$$(3.9) \quad \zeta(s, 1-a) = \sum_{j=0}^{\infty} \binom{-s}{j} \zeta(j+s)(-a)^j.$$

Substitution of (3.8) and (3.9) in (3.2) and (3.3), changing s in $1-s$, yields at once the expansions (1.18) and (1.19).

We shall also give an independent proof which makes no use of the relation of Hurwitz. We start from

$$\Gamma(s) F(x, s) = \int_0^{\infty} \frac{e^{2\pi x i}}{e^t - e^{2\pi x i}} t^{s-1} dt.$$

Since

$$\frac{e^{2\pi x i}}{e^t - e^{2\pi x i}} = \sum_{m=-\infty}^{\infty} \left\{ t^{-2(m+x)\pi i} \right\}^{-1}$$

we have

$$\begin{aligned} F(x, s) &= \sum_{m=-\infty}^{\infty} \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} dt}{t^{-2(m+x)\pi i}} = \\ &= \Gamma(1-s) \sum_{m=-\infty}^{\infty} \left\{ -2(m+x)\pi i \right\}^{s-1} = \\ &= \Gamma(1-s) \left\{ (-2\pi x i)^{s-1} + \sum_{m=1}^{\infty} \left\{ i^{s-1} \left(1 - \frac{x}{m}\right)^{s-1} + i^{-s+1} \left(1 + \frac{x}{m}\right)^{s-1} \right\} (2m\pi)^{s-1} \right\} = \\ &= \Gamma(1-s) (2\pi)^{s-1} \left\{ (-xi)^{s-1} + 2 \sum_{m=1}^{\infty} \sum_{j=0}^{\infty} \binom{s-1}{j} \cos \frac{1}{2} \pi (s+j-1) (ix)^j m^{s-j-1} \right\}, \end{aligned}$$

so that

$$F(x, s) = \Gamma(1-s) (2\pi)^{s-1} \left\{ (-xi)^{s-1} + 2 \sum_{j=0}^{\infty} \binom{s-1}{j} \sin \frac{1}{2} \pi (s+j) \zeta(j+1-s) (ix)^j \right\}.$$

By taking the real and the imaginary part the relations (1.18) and (1.19) follow at once.
